SOLUTION BOOKLET

Introduction to Methods in Operational Research

MADE BY

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My Story

- I am 25-year old guy from Serbia. I graduated with an MSc in Statistics and Data Science from KU Leuven — **as the only recipient of a full scholarship**. And earned **magna cum laude** honors.
- I have done hundreds of tutoring sessions, helping dozens of BBA students succeed, **nearly all of them passed their exams.**

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Contents

1 Chapter 1 - Indefinite Integrals **4**

Chapter 1 - Question 1 4

Chapter 1 - Question 2 5

Chapter 1 - Question 3 7

Chapter 1 - Question 4 9

Chapter 1 - Question 5 10

Chapter 1 - Indefinite Integrals

Question 1

Show that F is an anti-derivative of f.

(a) $F(x) = \dfrac{x^2}{x+1}, \quad f(x) = \dfrac{x^2+2x}{(x+1)^2}$

(b) $F(x) = x - \ln(1+e^x), \quad f(x) = \dfrac{1}{e^x+1}$

Tip: Verification via Differentiation

To show that $F(x)$ is an anti-derivative of $f(x)$, follow these steps:

1. **Differentiate $F(x)$:** Compute $F'(x)$ using the appropriate rules (Quotient Rule, Chain Rule, etc.).
2. **Simplify:** Algebraically manipulate $F'(x)$ to match the form of $f(x)$.
3. **Conclude:** If $F'(x) = f(x)$, then $F(x)$ is an anti-derivative of $f(x)$.

Solution 1: (a) $F(x) = \dfrac{x^2}{x+1}, \quad f(x) = \dfrac{x^2+2x}{(x+1)^2}$

The function f is defined for all $x \neq -1$. We apply the Quotient Rule:

$$\begin{aligned} F'(x) &= \frac{(2x)(x+1) - (x^2)(1)}{(x+1)^2} \\ &= \frac{2x^2 + 2x - x^2}{(x+1)^2} \\ &= \frac{x^2+2x}{(x+1)^2} \end{aligned}$$

Since $F'(x) = f(x)$, F is an anti-derivative of f.

Answer: $F'(x) = \frac{x^2+2x}{(x+1)^2}$

Solution 1: (b) $F(x) = x - \ln(1 + e^x), \quad f(x) = \frac{1}{e^x + 1}$

Differentiating $F(x)$ term by term:

$$\begin{aligned} F'(x) &= (x)' - \left(\ln(1 + e^x)\right)' \\ &= 1 - \frac{1}{1 + e^x} \cdot (1 + e^x)' \\ &= 1 - \frac{1}{1 + e^x} \cdot e^x \\ &= \frac{1 + e^x}{1 + e^x} - \frac{e^x}{1 + e^x} \end{aligned}$$

Simplifying the expression:

$$F'(x) = \frac{1 + e^x - e^x}{1 + e^x} = \frac{1}{1 + e^x}$$

Since $F'(x) = f(x)$, F is an anti-derivative of f.

Answer: $F'(x) = \frac{1}{e^x+1}$

Question 2

Show that:

(a) $$\int \frac{x^2\,dx}{\sqrt{1+x}} = \frac{2}{15}(3x^2 - 4x + 8)\sqrt{1+x} + C$$

(b) $$\int \frac{du}{2^u + 3} = \frac{u}{3} - \frac{1}{3\ln 2}\ln(2^u + 3) + C$$

Solution 2: (a) $\int \frac{x^2\,dx}{\sqrt{1+x}} = \frac{2}{15}(3x^2 - 4x + 8)\sqrt{1+x} + C$

Let $F(x) = \frac{2}{15}(3x^2 - 4x + 8)(1+x)^{1/2}$. We apply the Product Rule:

$$\begin{aligned}
F'(x) &= \frac{2}{15}\left[(6x-4)\sqrt{1+x} + (3x^2 - 4x + 8)\frac{1}{2\sqrt{1+x}}\right] \\
&= \frac{2}{15}\left[\frac{2(6x-4)(1+x) + (3x^2 - 4x + 8)}{2\sqrt{1+x}}\right] \\
&= \frac{1}{15\sqrt{1+x}}\left[2(6x^2 + 2x - 4) + 3x^2 - 4x + 8\right] \\
&= \frac{1}{15\sqrt{1+x}}\left[12x^2 + 4x - 8 + 3x^2 - 4x + 8\right] \\
&= \frac{15x^2}{15\sqrt{1+x}} = \frac{x^2}{\sqrt{1+x}}
\end{aligned}$$

Since $F'(x)$ matches the integrand, the equality is shown.

Answer: $F'(x) = \frac{x^2}{\sqrt{1+x}}$

Solution 2: (b) $\int \frac{du}{2^u + 3} = \frac{u}{3} - \frac{1}{3\ln 2}\ln(2^u + 3) + C$

Let $F(u) = \frac{u}{3} - \frac{1}{3\ln 2}\ln(2^u + 3)$. Differentiating with respect to u:

$$\begin{aligned} F'(u) &= \frac{1}{3} - \frac{1}{3\ln 2} \cdot \frac{1}{2^u + 3} \cdot (2^u \ln 2) \\ &= \frac{1}{3} - \frac{2^u \ln 2}{3\ln 2(2^u + 3)} \\ &= \frac{1}{3} - \frac{2^u}{3(2^u + 3)} \end{aligned}$$

Finding a common denominator:

$$F'(u) = \frac{(2^u + 3) - 2^u}{3(2^u + 3)} = \frac{3}{3(2^u + 3)} = \frac{1}{2^u + 3}$$

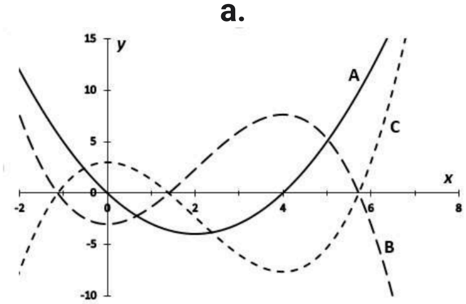
Since $F'(u)$ matches the integrand, the equality is shown.

Answer: $F'(u) = \frac{1}{2^u+3}$

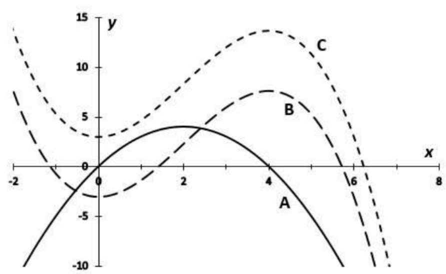
Question 3

In each of the following figures, three graphs are shown. If A is the graph of the function f, which of the other two graphs is the graph of an anti-derivative of f?

a.



b.



Tip: Relationship Between a Function and its Anti-derivative

To identify the graph of an anti-derivative F from the graph of f, follow these steps:

1. **Check the Zeroes:** When the graph of f is at zero (crosses the x-axis), the graph of F must have a local maximum or minimum (a flat turning point).
2. **Check the Sign:** If f is positive (above the x-axis), F must be increasing. If f is negative (below the x-axis), F must be decreasing.
3. **Check the Peaks:** When f has a peak or valley, F changes its curvature (inflection point).

Solution 3: (a) Identifying the anti-derivative for graph A

In figure (a), graph A represents the function f. We look at where graph A crosses the x-axis:

- Graph A is zero at $x \approx 0.5$ and $x = 4$.
- At $x \approx 0.5$, graph C has a local minimum (it turns upward).
- At $x = 4$, graph C has a local maximum (it turns downward).
- Graph B does not have turning points at these specific x-values.

Additionally, between $x \approx 0.5$ and $x = 4$, graph A is positive, and graph C is increasing during this entire interval.

Answer: Graph C

Solution 3: (b) Identifying the anti-derivative for graph A

In figure (b), graph A is a downward-opening parabola representing f.

- Graph A crosses the x-axis at $x = 0$ and $x = 4$.
- Both graph B and graph C have turning points at exactly $x = 0$ (local minimums) and $x = 4$ (local maximums).
- Between $x = 0$ and $x = 4$, graph A is positive, and both B and C are increasing.

Because B and C have the exact same shape and only differ by a vertical shift (a constant C), they both represent valid anti-derivatives of the same function f.

Answer: Both B and C

Question 4

Why is the formula $\int r f(x)\,dx = r \int f(x)\,dx$ not correct when $r = 0$?

Tip: Constants in Integration

When working with indefinite integrals and constants, keep these rules in mind:

1. **Integral of Zero:** The integral of zero is an arbitrary constant C, because the derivative of any constant is zero.
2. **Multiplication by Zero:** Multiplying any finished expression by zero results in exactly zero, which removes the constant of integration.

Solution 4

We compare the two sides of the equation when the constant r is zero:

- **Left-Hand Side:** The multiplication happens *inside* the integral before integration:

$$\int 0 \cdot f(x)\, dx = \int 0\, dx = C$$

  The result is an arbitrary constant C, which represents any real number.

- **Right-Hand Side:** The multiplication happens *after* the integration is performed:

$$0 \cdot \int f(x)\, dx = 0 \cdot (F(x) + C) = 0$$

  The result is exactly zero.

Since an arbitrary constant C is not necessarily equal to zero, the two sides are not equal. The formula fails in this specific case because it "loses" the constant of integration on the right side.

Answer: $C \neq 0$

Question 5

Calculate using integrals of basic functions and basic integration rules:

a. $\displaystyle\int \frac{1}{x^4}\,dx$

b. $\displaystyle\int \sqrt[3]{x}\,dx$

c. $\displaystyle\int (t+1)\sqrt{t}\,dt$

d. $\displaystyle\int (2^x+3^x)\,dx$

e. $\displaystyle\int \frac{2^x}{3^x}\,dx$

f. $\displaystyle\int \frac{x^3+x-1}{\sqrt[3]{x}}\,dx$

g. $\displaystyle\int (\sqrt{x}+1)(x-\sqrt{x})\,dx$

h. $\displaystyle\int \left(x^3-\frac{7}{x}+e^x\right)\,dx$

i. $\displaystyle\int \frac{\sqrt{t}-t+t^3}{\sqrt[3]{t}}\,dt$

j. $\displaystyle\int (x^e+e^x)\,dx$

k. $\displaystyle\int e^{\sqrt{2}}\,dx$

Tip: Basic Integration Rules

To solve these problems, we apply the basic building blocks of integration.

Power Rule	$\int x^n\,dx = \frac{x^{n+1}}{n+1}+C$	$n \neq -1$
Logarithmic	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$	
General Exponential	$\int a^x\,dx = \frac{a^x}{\ln a}+C$	

Solution 5: (a) $\displaystyle\int \frac{1}{x^4}\,dx$

Rewrite using a negative exponent:

$$\int x^{-4}\,dx$$

Apply the power rule $\int x^n\,dx = \frac{x^{n+1}}{n+1}+C$:

$$\frac{x^{-4+1}}{-4+1}+C = \frac{x^{-3}}{-3}+C = -\frac{1}{3x^3}+C$$

Answer: $-\frac{1}{3x^3} + C$

Solution 5: (b) $\int \sqrt[3]{x}\, dx$

Rewrite the radical as a fractional power:

$$\int x^{1/3}\, dx$$

Apply the power rule:

$$\frac{x^{1/3+1}}{1/3+1} + C \;=\; \frac{x^{4/3}}{4/3} + C \;=\; \frac{3}{4}x^{4/3} + C$$

Convert back to radical form:

$$\frac{3}{4}\sqrt[3]{x^4} + C \;=\; \frac{3}{4}x\sqrt[3]{x} + C$$

Answer: $\frac{3}{4}x\sqrt[3]{x} + C$

Tip: Addition, Subtraction, and Scaling

Integration follows linear properties, allowing you to break down complex expressions:

- **Sum and Difference Rule:** You can integrate terms separately:

$$\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

- **Constant Multiple Rule:** A constant factor can be moved outside the integral:

$$\int k \cdot f(x)\, dx = k \cdot \int f(x)\, dx$$

Tip: Combining Constants of Integration

When integrating a function with multiple terms, each term technically produces its own constant (C_1, C_2, C_3, etc.). However, since the sum or difference of any constant is still just an unknown constant, we always combine them into a single $+C$ at the very end.

$$\int (x+1)\,dx = \left(\frac{x^2}{2} + C_1\right) + (x + C_2) = \frac{x^2}{2} + x + C$$

Solution 5: (c) $\int (t+1)\sqrt{t}\,dt$

Rewrite $\sqrt{t}$ as $t^{1/2}$ and distribute into the parentheses:

$$\int (t \cdot t^{1/2} + 1 \cdot t^{1/2})\,dt = \int (t^{3/2} + t^{1/2})\,dt$$

Integrate term by term using the power rule:

$$\frac{t^{3/2+1}}{3/2+1} + \frac{t^{1/2+1}}{1/2+1} + C = \frac{t^{5/2}}{5/2} + \frac{t^{3/2}}{3/2} + C$$

Simplify the fractions:

$$\frac{2}{5}t^{5/2} + \frac{2}{3}t^{3/2} + C = \frac{2}{5}t^2\sqrt{t} + \frac{2}{3}t\sqrt{t} + C$$

Answer: $\frac{2}{5}t^2\sqrt{t} + \frac{2}{3}t\sqrt{t} + C$

Solution 5: (d) $\int (2^x + 3^x)\, dx$

Apply the exponential rule $\int a^x\, dx = \frac{a^x}{\ln a} + C$ to each term:

$$\int 2^x\, dx + \int 3^x\, dx = \frac{2^x}{\ln 2} + \frac{3^x}{\ln 3} + C$$

Answer: $\frac{2^x}{\ln 2} + \frac{3^x}{\ln 3} + C$

Solution 5: (e) $\int \frac{2^x}{3^x}\, dx$

Rewrite using the property $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$:

$$\int \left(\frac{2}{3}\right)^x\, dx$$

Apply the exponential rule with $a = \frac{2}{3}$:

$$\frac{(2/3)^x}{\ln(2/3)} + C = \frac{1}{\ln(2/3)} \cdot \frac{2^x}{3^x} + C$$

Answer: $\frac{1}{\ln(2/3)} \frac{2^x}{3^x} + C$

Solution 5: (f) $\int \frac{x^3 + x - 1}{\sqrt[3]{x}}\,dx$

Divide each term in the numerator by $x^{1/3}$:

$$\int \left(\frac{x^3}{x^{1/3}} + \frac{x^1}{x^{1/3}} - \frac{1}{x^{1/3}}\right) dx \;=\; \int (x^{8/3} + x^{2/3} - x^{-1/3})\,dx$$

Apply the power rule to each term:

$$\frac{x^{11/3}}{11/3} + \frac{x^{5/3}}{5/3} - \frac{x^{2/3}}{2/3} + C \;=\; \frac{3}{11}x^{11/3} + \frac{3}{5}x^{5/3} - \frac{3}{2}x^{2/3} + C$$

Factor out whole powers:

$$\frac{3}{11}x^3\sqrt[3]{x^2} + \frac{3}{5}x\sqrt[3]{x^2} - \frac{3}{2}\sqrt[3]{x^2} + C$$

Answer: $\frac{3}{11}x^3\sqrt[3]{x^2} + \frac{3}{5}x\sqrt[3]{x^2} - \frac{3}{2}\sqrt[3]{x^2} + C$

Solution 5: (g) $\int (\sqrt{x} + 1)(x - \sqrt{x})\,dx$

Expand the expression:

$$\int (\sqrt{x} \cdot x - \sqrt{x} \cdot \sqrt{x} + 1 \cdot x - 1 \cdot \sqrt{x})\,dx$$

$$\int (x^{3/2} - x + x - x^{1/2})\,dx \;=\; \int (x^{3/2} - x^{1/2})\,dx$$

Integrate using the power rule:

$$\frac{x^{5/2}}{5/2} - \frac{x^{3/2}}{3/2} + C \;=\; \frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2} + C$$

Answer: $\frac{2}{5}x^2\sqrt{x} - \frac{2}{3}x\sqrt{x} + C$

Tip: Special Cases: Integrals of e^x and Constants

Two fundamental integration rules to remember are:

- **The Exponential Rule:** The integral of e^x is the most straightforward as it remains unchanged:

$$\int e^x \, dx = e^x + C$$

- **The Constant Rule:** The integral of a constant k (any real number) is that constant multiplied by the variable:

$$\int k \, dx = kx + C$$

Solution 5: (h) $\int \left(x^3 - \frac{7}{x} + e^x\right) dx$

Integrate term by term:

$$\int x^3 \, dx - 7 \int \frac{1}{x} \, dx + \int e^x \, dx$$

Apply the power rule, the log rule, and the exponential rule:

$$\frac{x^4}{4} - 7 \ln |x| + e^x + C$$

Answer: $\frac{x^4}{4} - 7 \ln |x| + e^x + C$

Solution 5: (i) $\int \frac{\sqrt{t} - t + t^3}{\sqrt[3]{t}}\, dt$

Rewrite radicals as fractional powers and divide:

$$\int \left(\frac{t^{1/2}}{t^{1/3}} - \frac{t^1}{t^{1/3}} + \frac{t^3}{t^{1/3}} \right) dt \;=\; \int (t^{1/6} - t^{2/3} + t^{8/3})\, dt$$

Integrate each term:

$$\frac{t^{7/6}}{7/6} - \frac{t^{5/3}}{5/3} + \frac{t^{11/3}}{11/3} + C \;=\; \frac{6}{7}t^{7/6} - \frac{3}{5}t^{5/3} + \frac{3}{11}t^{11/3} + C$$

Answer: $\frac{6}{7}t\sqrt[6]{t} - \frac{3}{5}t\sqrt[3]{t^2} + \frac{3}{11}t^3\sqrt[3]{t^2} + C$

Solution 5: (j) $\int (x^e + e^x)\, dx$

We treat the first term using the Power Rule for a constant exponent ($e \approx 2.718$) and the second term using the Exponential Rule for a constant base:

$$\begin{aligned} \int (x^e + e^x)\, dx &= \int x^e\, dx + \int e^x\, dx \\ &= \frac{x^{e+1}}{e+1} + e^x + C \end{aligned}$$

Answer: $\frac{x^{e+1}}{e+1} + e^x + C$

Solution 5: (k) $\int e^{\sqrt{2}}\,dx$

Since $e^{\sqrt{2}}$ is a constant (no x variable in the base or exponent), the integral is simply the constant times x:

$$e^{\sqrt{2}} \cdot \int 1\,dx \;=\; e^{\sqrt{2}}x + C$$

Answer: $e^{\sqrt{2}}x + C$